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On the interrelation between the solutions of the mKP and KP equations via the Miura transformation

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Abstract. It is shown that the classes of exact solutions with functional parameters and rational solutions of the modified Kadomtsev-Petviashvili (mKP) equation and the Kadomtsev-Petviashvili (KP) equation are connected by the (2+1)-dimensional Miura transformation. The correspondence between more particular classes of solutions of the mKP and the KP equations via the Miura transformation is established.

1. Introduction

The Miura transformation [1] between the modified Korteweg-de Vries (mKdV) and the Korteweg-de Vries (KdV) equations has played an important role both in the discovery of the inverse scattering transform (IST) method [2] and in the further understanding of the properties of these equations (e.g. see [3-5]). It reveals a deep interrelation between the algebraic properties of the mKdV and KdV equations and their *hierarchies* [3-5].

The (2+1)-dimensional integrable generalizations of the KdV and mKdV equations are given by the well known Kadomtsev-Petviashvili (KP) equation

$$U_t + U_{xxx} + 6UU_x + 3\sigma^2\partial_x^{-1}U_{yy} = 0 \tag{1}$$

and the modified KP (mKP) equation

$$V_t + V_{xxx} - 3\sigma^2(\tfrac{1}{2}V^2V_x - V_x\partial_x^{-1}V_y + \partial_x^{-1}V_{yy}) = 0 \tag{2}$$

which has been introduced within the different approaches in [6, 7]. Here $\sigma^2 = \pm 1$. The mKP and KP equations are connected by the (2+1)-dimensional generalization of the Miura transformation. Namely, if the function V obeys the mKP equation (2) then the function

$$U = -\tfrac{1}{2}\sigma^2\partial_x^{-1}V_y - \tfrac{1}{2}\sigma V_x - \tfrac{1}{4}\sigma^2V^2 \tag{3}$$

obeys the KP equation [6, 7]. Similar to the (1+1)-dimensional case the Miura transformation (3) deeply interrelates the algebraic structures associated with the mKP and KP equations [8].

In the present paper the properties of the classes of exact solutions of the mKP and the KP equations under the Miura map (3) are studied. We will show that the solutions of the mKP equation (2) with functional parameters are converted under the map (3) into the solutions of the KP equation (1) with, essentially, the same functional parameters. The Miura map (3) transforms the rational solutions of the mKP equation into

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the rational solutions of the KP equation. The interrelation between the plane solitons, decreasing and plane lumps, non-singular and singular, real and complex solutions of the mKP and KP equations are established. Both of the cases $\sigma^2=-1$ and $\sigma^2=+1$ are considered.

2. General formulae

The classes of exact solutions of the KP equation, including solutions with functional parameters, plane solutions, singular rational solutions and lumps, are well known (e.g. see [3, 5]). For the mKP equation similar classes of exact solutions have been constructed recently in [9]. All the details about the properties of the solutions of the KP and mKP equations can be found in these papers.

The correspondence between the solutions of the mKP and KP equations can be established directly in terms of V and U with the use of the map (3). But it is more convenient and transparent to do this using the mKP eigenfunction χ. The first linear problem for the mKP equation (2) is of the form [9]

$$\sigma\psi_y+\psi_{xx}+\sigma V\psi_x=0. \tag{4}$$

The eigenfunction $\chi(x, y, t; \lambda)$ relevant for the formulation and solving the inverse spectral problem for the mKP equation is introduced via [9]

$$\psi \overset{\text{def}}{=} \chi \exp\left(\frac{\mathrm{i}x}{\lambda}+\frac{y}{\sigma\lambda^2}\right). \tag{5}$$

It obeys the equation

$$\sigma\chi_y+\chi_{xx}+\frac{\mathrm{i}\sigma V}{\lambda}\chi+\sigma V\chi_x+\frac{2\mathrm{i}}{\lambda}\chi_x=0. \tag{6}$$

The function $\chi(x, y, t; \lambda)$ can be canonically normalized ($\chi \xrightarrow[\lambda\to\infty]{} 1$) and the reconstruction formula for the potential V is of the form [9]

$$V(x, y, t)=-\frac{2}{\sigma}\frac{\partial}{\partial x}\ln\chi_0 \tag{7}$$

where χ_0 is the first of the coefficients of the Taylor series expansion of χ near the origin:

$$\chi(x, y, t; \lambda)=\chi_0(x, y, t)+\lambda\chi_1(x, y, t)+\lambda^2\chi_2(x, y, t)+\ldots. \tag{8}$$

The equations of the inverse problem for (6) have the wide classes of exact solutions which give rise via (7) to the classes of exact solutions of the mKP equations [9].

Formula (7) is very convenient for the analysis of the Miura map (3). Substituting (7) into (3), one easily gets

$$U=-\frac{\sigma(\chi_0^{-1})_y+(\chi_0^{-1})_{xx}}{\chi_0^{-1}} \tag{9}$$

which clearly demonstrates that U is, indeed, the solution of the KP equation. Moreover, using the relation

$$\sigma\chi_{0y}+\chi_{0xx}+2\mathrm{i}\chi_{1x}+\mathrm{i}\sigma V\chi_1+\sigma V\chi_{0x}=0 \tag{10}$$

which arises from the substitution of expansion (8) into (6), one can transform the RHS of (9) into the simpler form. Namely,

$$U=-2\mathrm{i}\frac{\partial}{\partial x}\left(\frac{\chi_1}{\chi_0}\right). \tag{11}$$

This formula gives us a simple way for calculating the solutions of the KP equation using the known functions χ_0 and χ_1 for the mKP equation.

It is easy to see that the Miura map (3) transforms the real-valued solutions of the mKP-I equation ($\sigma=\mathrm{i}$) into the complex-valued solutions of the KP equation ($\sigma=\mathrm{i}$) while in the case $\sigma=1$ the Miura transformation (3) connects the real-valued solutions of the mKP and KP equations. The Miura transformation (9) maps the non-singular solutions of the mKP equation into the non-singular solutions of the KP equation.

3. Correspondence between the solutions with functional parameters and rational solutions

The most general classes of exact solutions of the mKP equations include the solutions with functional parameters and rational in x-, y-, t-solutions. Here we will consider the correspondence between them.

The solutions of the mKP equation with functional parameters are of the form [9]

$$V=-\frac{2}{\sigma}\frac{\partial}{\partial x}\ln(\det\tilde{A}A^{-1}) \tag{12}$$

where

$$\begin{aligned} A_{kl}&=\delta_{kl}-\frac{1}{2\mathrm{i}}\partial_x^{-1}\xi_{kx}\eta_l \\ \tilde{A}_{kl}&=\delta_{kl}+\frac{1}{2\mathrm{i}}\partial_x^{-1}\xi_k\eta_{lx} \end{aligned} \tag{13}$$

and

$$\chi_0=\det(\tilde{A}A^{-1}) \qquad \chi_1=\operatorname{tr}(BA^{-1}) \tag{14}$$

where

$$B_{kl}=\mathrm{i}\partial_x\tilde{A}_{kl}=\tfrac{1}{2}\xi_k\eta_{lx}. \tag{15}$$

Here $\xi_k(x,y,t)$ and $\eta_l(x,y,t)$ are the solutions of the linearized mKP equation

$$\xi_t+\xi_{xxx}+3\sigma^2\partial_x^{-1}\xi_{yy}=0 \tag{16}$$

of the form

$$\begin{aligned} \xi_k(x,y,t)&=\int\!\!\int_{\mathbb{C}}\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}\,f_k(\lambda,\bar{\lambda})\exp\left(\frac{\mathrm{i}x}{\lambda}+\frac{y}{\sigma\lambda^2}+\frac{4\mathrm{i}t}{\lambda^3}\right) \\ \eta_l(x,y,t)&=\int\!\!\int_{\mathbb{C}}\mathrm{d}\lambda\wedge\mathrm{d}\bar{\lambda}\,g_k(\lambda,\bar{\lambda})\exp\left(\frac{\mathrm{i}x}{\lambda}+\frac{y}{\sigma\lambda^2}+\frac{4\mathrm{i}t}{\lambda^3}\right) \end{aligned} \tag{17}$$

where $f_k(\lambda, \bar{\lambda})$ and $g_k(\lambda, \bar{\lambda})$ are arbitrary complex functions. The integral ∂_x^{-1} in (13), (14) is defined in such a way that RHSs of (13) and (14) exist.

Now let us apply the Miura transformation to the solutions (12). Formula (11) gives

$$U = 2\frac{\partial}{\partial x}\frac{\operatorname{tr}(\tilde{A}_x A^{-1})}{\det(\tilde{A}A^{-1})}. \tag{18}$$

Note that the matrices $\tilde{A}_x A^{-1}$, $1 - A\tilde{A}^{-1} - \tilde{A}_x\tilde{A}^{-1}$ and $1 - A\tilde{A}^{-1}$ have rank one. For rank one matrices one has the well known identity

$$\det(1+F) = 1 + \operatorname{tr} F. \tag{19}$$

Using (19) and another well known matrix identity,

$$\frac{\partial}{\partial x}\ln\det F = \operatorname{tr}(F_x F^{-1}) \tag{20}$$

one gets

$$\begin{aligned} \frac{\operatorname{tr}(\tilde{A}_x A^{-1})}{\det(\tilde{A}A^{-1})} &= \frac{\det(1+\tilde{A}_x A^{-1}) - 1}{\det(\tilde{A}A^{-1})} \\ &= \det[1 - (1 - A\tilde{A}^{-1} - \tilde{A}_x\tilde{A}^{-1})] - \det[1 - (1 - A\tilde{A}^{-1})] \\ &= \operatorname{tr}(\tilde{A}_x\tilde{A}^{-1}) = \frac{\partial}{\partial x}\ln\det\tilde{A}. \end{aligned} \tag{21}$$

So we finally obtain the solutions of the KP equation

$$U = 2\frac{\partial^2}{\partial x^2}\ln\det\tilde{A} \tag{22}$$

with the functional parameters, where $\tilde{A}$ is given by (13). Formula (22) after the identification

$$\xi_l^{\mathrm{KP}} = \xi_l \qquad \eta_k^{\mathrm{KP}} = \eta_{kx} \tag{23}$$

exactly coincides with the known formula for the solutions of the KP equation with functional parameters (e.g. see [3]).

The linear parts of the KP and the mKP equations coincide. So, the same set (up to the change (23)) of solutions of the linear equation (16) parametrize the classes of exact solutions (12) and (22) of the mKP and KP equations, respectively. The Miura map (3) connects these classes of solutions without, in essence, changing the parameters ξ_l and η_k.

A similar situation takes place for the rational solutions. The general rational solutions of the mKP equation are of the form [9]

$$V = -\frac{2}{\sigma}\frac{\partial}{\partial x}\ln\det(\tilde{A}A^{-1}) \tag{24}$$

where

$$\begin{aligned} A_{kl} &= \delta_{kl}\left(x - \frac{2\mathrm{i}y}{6\lambda_k} + \frac{12t}{\lambda_k^2} + \gamma_k\right) + (1-\delta_{kl})\frac{\mathrm{i}\lambda_l^2}{\lambda_k - \lambda_l} \\ \tilde{A}_{kl} &= A_{kl} + \mathrm{i}\lambda_l \end{aligned} \tag{25}$$

and

$$\chi_0 = \det(\tilde{A}A^{-1}) \qquad \chi_1 = \mathrm{i}\,\mathrm{tr}(\mathscr{E}A^{-1}) \tag{26}$$

where $\mathscr{E}_{kl} = 1$ $(k, l = 1, \ldots, N)$.

The Miura map (3) converts the solutions (24) into the following solutions of the KP equation:

$$U = 2\frac{\partial}{\partial x}\frac{\mathrm{tr}(\mathscr{E}A^{-1})}{\det(\tilde{A}A^{-1})}. \tag{27}$$

Taking into account that the matrices $1 - A\tilde{A}^{-1} - \mathscr{E}\tilde{A}^{-1}$ and $1 - A\tilde{A}^{-1}$ have rank one, and using the identities (19), (20), similar to the previous case, one gets

$$U = 2\frac{\partial}{\partial x}\mathrm{tr}(\mathscr{E}\tilde{A}^{-1}). \tag{28}$$

Finally, using the properties of matrix $\tilde{A}$ (in particular, $(\tilde{A}_{pq})_x = \delta_{pq}$), one obtains

$$U = 2\frac{\partial^2}{\partial x^2}\ln\det\tilde{A} \tag{29}$$

where $\tilde{A}$ is given by (25), i.e.

$$\tilde{A}_{kl} = \delta_{kl}\left(x - \frac{2\mathrm{i}y}{\sigma\lambda_k} + \frac{12t}{\lambda_k^2} + \gamma_k\right) + \mathrm{i}(1-\delta_{kl})\frac{1}{\lambda_l^{-1} - \lambda_k^{-1}}. \tag{30}$$

The formulae (29), (30) coincide with the well known formulae for the rational solutions of the KP equation (up to $\lambda_k \to \lambda_k^{-1}$) (see [3]).

4. Correspondence between particular classes of solutions: the $\sigma = \mathrm{i}$ case

Now we will consider more particular classes of solutions, including the real and non-singular solutions.

We start with the case $\sigma = \mathrm{i}(\sigma^2 = -1)$. The real non-singular plane solitons of the mKP-I equation are given by formula (12) with [9]

$$\xi_l = -2\mathrm{i}\,R_l\exp(F(\lambda_l)) \qquad \eta_k = -2\mathrm{i}\exp(-F(\bar{\lambda}_k)) \tag{31}$$

where

$$\mathrm{Im}\,R_l = 0 \qquad \text{and} \qquad F(\lambda) \stackrel{\text{def}}{=} \frac{\mathrm{i}x}{\lambda} - \frac{\mathrm{i}y}{\lambda^2} + \frac{4\mathrm{i}t}{\lambda^3}.$$

It is not difficult to show, using (22), that the real non-singular plane solitons of the mKP-I equations are transformed into complex non-singular plane solitons of the KP-I equation. In particular, the simplest mKP-I soliton [9]

$$V = -\frac{8(\lambda_I/|\lambda|^2)\,\mathrm{sign}\,R}{\mathrm{e}^{2f} + [\mathrm{e}^{-f} + (\lambda_R/\lambda_I)(\mathrm{sign}\,R)\,\mathrm{e}^{f}]^2} \tag{32}$$

is converted into the complex non-singular soliton of the KP-I equation:

$$U = \frac{8(\mathrm{sign}\,R)\lambda\lambda_I}{|\lambda|^4[\mathrm{e}^{-f} + (\lambda/\lambda_I)\mathrm{e}^{f}\,\mathrm{sign}\,R]^2} \tag{33}$$

where

$$2f = \mathrm{i}x(\lambda^{-1} - \bar{\lambda}^{-1}) - \mathrm{i}y(\lambda^{-2} - \bar{\lambda}^{-2}) + 4\mathrm{i}t(\lambda^{-3} - \bar{\lambda}^{-3}) + \ln|R|. \tag{34}$$

The well known plane real-valued non-singular solitons of the KP-I are connected via the Miura transformation (3) with the complex non-singular plane solitons of the mKP-I equation. In particular, the well known one-soliton solution of the KP-I equation

$$U = \frac{2\lambda_{\mathrm{I}}^2}{|\lambda|^4} \frac{1}{\cosh^2 f} \tag{35}$$

is obtained from the complex non-singular soliton

$$V = -\frac{4\lambda_{\mathrm{I}}^2}{\lambda|\lambda|^2} \frac{1}{[\mathrm{e}^{-f} + (\bar{\lambda}/\lambda)\,\mathrm{e}^{f}]\cosh f} \tag{36}$$

of the mKP-I equation, where f is given by (34).

The solutions of the mKP-I equation of the breather type, constructed in [9] are converted by the Miura transformation into the periodic in x- or y-solutions of the KP-I equation. In particular, the complex breather-type solution of the mKP-I equation which can be obtained by the technique of the work [9] is of the form

$$V = 2\mathrm{i}\frac{\partial}{\partial x} \ln \frac{(1 - \mathrm{e}^{f}\sin\varphi/\lambda_{\mathrm{R}})^2 + (\mathrm{e}^{2f}/\lambda_{\mathrm{R}}^2\lambda_{\mathrm{I}}^2)(\lambda_{\mathrm{R}}^2 + \lambda_{\mathrm{I}}^2\cos^2\varphi)}{1 + (2\,\mathrm{e}^{f}\bar{\lambda}^2/\lambda_{\mathrm{R}}|\lambda|^2)\sin\varphi + \bar{\lambda}^2\,\mathrm{e}^{2f}/\lambda_{\mathrm{R}}^2\lambda_{\mathrm{I}}^2} \tag{37}$$

where

$$\begin{aligned} f &= -\frac{4\lambda_{\mathrm{R}}\lambda_{\mathrm{I}}}{|\lambda|^4} y + \ln|\lambda R| \\ \varphi &= \frac{2\lambda_{\mathrm{R}}x}{|\lambda|^4} + \frac{8(\lambda_{\mathrm{R}}^3 - 3\lambda_{\mathrm{R}}\lambda_{\mathrm{I}}^2)t}{|\lambda|^6} + \arg(R\lambda) \end{aligned} \tag{38}$$

and R, some complex constant, is transformed into the following real solution of the KP-I equation:

$$U = 2\frac{\partial^2}{\partial x^2} \ln\left[\left(1 - \frac{\mathrm{e}^{f}\sin\varphi}{\lambda_{\mathrm{R}}}\right)^2 + \frac{\mathrm{e}^{2f}}{\lambda_{\mathrm{R}}^2\lambda_{\mathrm{I}}^2}(\lambda_{\mathrm{R}}^2 + \lambda_{\mathrm{I}}^2\cos^2\varphi)\right]. \tag{39}$$

The solution (39) is the real-valued, non-singular solution of the KP-I equation decreasing at $y \to \pm\infty$, and has a periodic wave character in x, t.

Another complex breather-type solution of the mKP-I equation which can be obtained by the technique of [9] is of the form

$$\begin{aligned} V = 2\mathrm{i}\frac{\partial}{\partial x} \ln\Bigg\{ & \left(1 + a\,\mathrm{e}^{f}\cos\varphi + \frac{(\nu_1 + \nu_2)^2}{16\nu_1\nu_2} a^2\,\mathrm{e}^{2f}\right) \\ & \times \left[1 + \frac{a}{2}\mathrm{e}^{f}\left(\frac{\nu_2}{\nu_1}\mathrm{e}^{\mathrm{i}\varphi} + \frac{\nu_1}{\nu_2}\mathrm{e}^{-\mathrm{i}\varphi}\right) + \frac{a^2}{4}\mathrm{e}^{2f}\right]^{-1}\Bigg\} \end{aligned} \tag{40}$$

where

$$f(x,t) = x\left(\frac{1}{\nu_1} - \frac{1}{\nu_2}\right) - 4t\left(\frac{1}{\nu_1^3} - \frac{1}{\nu_2^3}\right)$$

$$\varphi(y) = y\left(\frac{1}{\nu_1^2} - \frac{1}{\nu_2^2}\right)$$

and a, some real constant, is converted by the Miura transformation into

$$U(x, y, t)=2\frac{\partial^2}{\partial x^2}\ln\left(1+a\,\mathrm{e}^f\cos\varphi+\frac{a^2}{16}\frac{(\nu_1+\nu_2)^2}{\nu_1\nu_2}\mathrm{e}^{2f}\right). \tag{41}$$

This real, non-singular, periodic in y and soliton type in the (x, t) solution of the KP-I equation has been found in [10].

5. The correspondence between the lumps ($\sigma=\mathrm{i}$)

The KP-I equation possesses real decreasing lumps (see [3, 5]) while the mKP-I equation has both real decreasing lumps and real plane lumps [9].

(i) The real decreasing lumps of the mKP-I equation are given by the formula (24) with [9]

$$\begin{gathered} N=2n \qquad \lambda_{n+i}=\bar{\lambda}_i \\ \gamma_i=-\frac{\mathrm{i}\lambda_i}{2}+c_i \qquad \gamma_{n+i}=-\mathrm{i}\frac{\bar{\lambda}_i}{2}+\bar{c}_i \end{gathered} \tag{42}$$

where $\lambda_i\ (i=1,2,\ldots,n)$ are arbitrary isolated points outside the real axis and c_i are arbitrary constants.

It is not difficult to see that the corresponding rational solutions (29), (30) of the KP-I equation are complex and non-singular. In particular, the real non-singular decreasing lump of the mKP-I equation [9]

$$V=\frac{2\lambda\bar{X}^2+2\bar{\lambda}X^2-|\lambda|^2\lambda_{\mathrm{R}}^3/\lambda_{\mathrm{I}}^2}{(|X|^2+|\lambda|^2\lambda_{\mathrm{R}}^2/4\lambda_{\mathrm{I}}^2)^2+[(\lambda/2)\bar{X}+(\bar{\lambda}/2)X]^2} \tag{43}$$

where

$$X=x-\frac{2y}{\lambda}+\frac{12t}{\lambda^2}+c$$

$$c=c_{\mathrm{R}}+\mathrm{i}c_{\mathrm{I}}$$

is converted into the complex non-singular decreasing rational solution

$$U=\frac{|\lambda|^2\lambda_{\mathrm{R}}^2/\lambda_{\mathrm{I}}^2-2X^2-2\bar{X}^2+2\lambda_{\mathrm{R}}^2-2\mathrm{i}\bar{\lambda}\bar{X}-2\mathrm{i}\lambda X}{\{|X|^2+|\lambda|^2\lambda_{\mathrm{R}}^2/4\lambda_{\mathrm{I}}^2+\mathrm{i}[(\lambda/2)\bar{X}+(\bar{\lambda}/2)X]\}^2} \tag{44}$$

of the KP-I equation.

(ii) The real plane lumps of the mKP-I equation are given by (24), (25) where [9]

$$\operatorname{Im}\lambda_i=0 \qquad \gamma_i=-\frac{\mathrm{i}\lambda_i}{2}+c_i \qquad \operatorname{Im}c_i=0 \qquad (i=1,\ldots,N).$$

They are mapped by the Miura transformation (3) into the complex plane non-singular rational solutions of the KP-I equation. For instance, the simplest plane lump of the mKP-I equation

$$V=\frac{2\alpha}{(x-2y/\alpha+12t/\alpha^2+x_0)^2+\alpha^2/4} \tag{45}$$

where α is an arbitrary real constant is converted to the complex plane non-singular lump:

$$U=2\frac{\alpha^2/4-(x-2y/\alpha+12t/\alpha^2+x_0)^2+\alpha\mathrm{i}(x-2y/\alpha+12t/\alpha^2+x_0)}{[(x-2y/\alpha+12t/\alpha^2+x_0)^2+\alpha^2/4]^2}. \tag{46}$$

(iii) Finally one can show that the real decreasing lumps of the KP-I equations [3, 5] are obtained by the Miura transformation (3) from the complex rational non-singular solutions of the mKP-I equation. In particular, the complex solution

$$V=2\mathrm{i}\left(\frac{X+\bar{X}}{|X|^2+|\lambda|^4/4\lambda_{\mathrm{I}}^2}-\frac{X+\bar{X}-2\mathrm{i}\lambda_{\mathrm{R}}}{|X|^2+|\lambda|^4/4\lambda_{\mathrm{I}}^2-|\lambda|^2-\mathrm{i}(\lambda\bar{X}+\bar{\lambda}X)}\right) \tag{47}$$

where

$$X=x-\frac{2y}{\lambda}+\frac{12t}{\lambda^2}+c$$

of the mKP-I equation is transformed into

$$U=\frac{|\lambda|^4/\lambda_{\mathrm{I}}^2-2X^2-2\bar{X}^2}{(|X|^2+|\lambda|^4/4\lambda_{\mathrm{I}}^2)^2} \tag{48}$$

that is, the well known real decreasing lump of the KP-I equation [3, 5].

6. The $\sigma=1$ case

In this case the real-valued solutions of the mKP-II equation are transformed by the Miura map into the real-valued solutions of the KP-II equation.

(i) The real plane solitons of the mKP-II equation are given by the formulae (12), (13) with

$$\begin{aligned}\xi_l&=-2\mathrm{i}R_l\exp(F(\mathrm{i}\alpha_l))\\ \eta_l&=-2\beta_l^{-1}\exp(-F(\mathrm{i}\beta_l))\end{aligned} \tag{49}$$

where R_l, α_l, β_l are arbitrary real constants. It is easy to see that the corresponding solutions of the KP-II equation are given by (22) with

$$\tilde{A}_{mn}=\delta_{mn}-\frac{2R_n\exp[x(\alpha_n^{-1}-\beta_m^{-1})-y(\alpha_n^{-2}-\beta_m^{-2})-4t(\alpha_n^{-3}-\beta_m^{-3})]}{\beta_n^2(\alpha_n^{-1}-\beta_n^{-1})} \tag{50}$$

which exactly coincides with the formula for the multisoliton solutions of the KP-II equation (with $\alpha_n^{-1}\to\lambda_n$) [3]. In particular, the simplest real soliton of the mKP-II equation [9]

$$V=-\frac{2(\alpha-\beta)^2}{\alpha\beta^3}\frac{\varepsilon}{(\mathrm{e}^{-f}-(\alpha/\beta)\varepsilon\,\mathrm{e}^{f})(\mathrm{e}^{-f}-\varepsilon\,\mathrm{e}^{f})} \tag{51}$$

where

$$\begin{aligned}2f&=(\alpha^{-1}-\beta^{-1})x-(\alpha^{-2}-\beta^{-2})y-4t(\alpha^{-3}-\beta^{-3})+\ln 2\left|\frac{R}{\beta-\alpha}\right|\\ \varepsilon&=\mathrm{sign}\left(\frac{R}{\beta-\alpha}\right)\end{aligned} \tag{52}$$

is converted into the well known real plane soliton of the KP-II equation:

$$U=\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2\cosh^{-2}\left(\frac{\tilde{f}}{2}\right) \tag{53}$$

where

$$\tilde{f}=x(\alpha^{-1}-\beta^{-1})-y(\alpha^{-2}-\beta^{-2})-4t(\alpha^{-3}-\beta^{-3})+\ln\frac{2R\alpha}{\beta(\alpha-\beta)}. \tag{54}$$

The function (53) is non-singular not only for those values of parameters $\alpha, \beta, \varepsilon$ ($\varepsilon<0$, $\alpha/\beta>0$) as for the mKP-II plane soliton (51) but also for $\varepsilon>0$, $\alpha/\beta<0$ for which the soliton (51) is the singular one. The properties of the KP-II plane solitons are quite different in these two cases. Namely, the soliton (53) at $\varepsilon>0$, $\alpha/\beta<0$ (type I) possesses at $\alpha=-\beta$ the non-trivial $(1+1)$-dimensional limit

$$U_{\mathrm{KdV}}=\frac{2}{\alpha^2}\cosh^{-2}\left(\frac{\varphi}{2}\right) \qquad \varphi=\tilde{f}|_{\alpha=-\beta} \tag{55}$$

that is, the standard KdV soliton, while at $\varepsilon<0$, $\alpha/\beta>0$ (type II) the solution (53) has a trivial $(1+1)$-dimensional limit $U|_{\alpha=\beta}=0$.

So the Miura transformation (3) maps the bounded plane solitons of the mKP-II equation into the type II (pure $(2+1)$-dimensional) plane solitons of the KP-II equation and the singular plane soliton of the mKP-II equation into the standard (type I) plane soliton of the KP-II equation.

This last property of the map (3) is similar to the property of the $(1+1)$-dimensional Miura map $U=-\frac{1}{2}V_x-\frac{1}{4}V^2$ which, as has been shown in [11], does not interrelate the rapidly decaying smooth solutions of the mKdV and KdV equations. This is quite clear from the consideration of the $(1+1)$-dimensional limit of the $(2+1)$-dimensional case. Indeed, the $(1+1)$-dimensional limit of the solution (51) $(\alpha=-\beta)$ looks like

$$V_{\mathrm{mKdV}}=\frac{4\varepsilon}{\alpha}\sinh^{-1}2\phi \qquad \phi=f|_{\alpha=-\beta} \tag{56}$$

that is, the singular solution of the mKdV equation while the corresponding limit of the solution (53) $(\alpha=-\beta)$ is given by (55). So the $(1+1)$-dimensional Miura transformation maps the singular solutions of the mKdV equation into the soliton of the KdV equation.

A similar situation takes place for general multisoliton solutions of the mKP-II and KP-II equations.

(ii) The rational solutions of the mKP-II equation are real-valued in the two cases [9]

(*a*) $N=2n \qquad \lambda_{k+n}=\bar{\lambda}_k \qquad \gamma_{k+n}=\bar{\gamma}_k \qquad (k=1,\ldots,n)$

(*b*) arbitrary $N \qquad \lambda_k=\mathrm{i}\alpha_k \qquad (\mathrm{Im}\,\alpha_k=0) \qquad \gamma_k=\bar{\gamma}_k.$

However, all these rational solutions of the mKP-II equation are singular.

They remain singular after the Miura transformation. In particular, the simplest singular plane lumps of the mKP-II equation

$$V=\frac{2\alpha}{\alpha^2/4-(x+2y/\alpha-12t/\alpha^2+x_0)^2} \tag{57}$$

where α is an arbitrary real constant is transformed into the solution

$$U=-\frac{2}{(x+2y/\alpha-12t/\alpha^2-\alpha/2+x_0)^2}. \tag{58}$$

This is the well known singular solution of the KP-II equation [3, 5]. Note that the singularity line $x=-2y/\alpha+12t/\alpha^2+\alpha/2-x_0$ of the solution (58) coincides with a singularity line of the solution (57).

A similar situation also takes place in the general case. Comparing (24) and (29), we see that the singularities of the solutions of the KP-II equation are defined by the zeros of det A and that these singularities present only part of the singularities of the solution (24) of the mKP-II equation.

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